Stochastic resonance via dynamical symmetry breaking in a modulated bistable potential

Subir K. Sarkar* and Debashish Bose†
School of Physical Sciences, Jawaharlal Nehru University, New Delhi 110 067, India
(Received 5 May 1998)

We demonstrate the possibility of observing stochastic resonance in the dynamics of the equation $\dot{x}(t)=x-x^3+\alpha\cos\omega t+$noise when the amplitude (α) of modulation is substantially higher than the static threshold of dynamical hysteresis. This is done by utilizing the phenomenon of dynamical symmetry breaking in which the noiseless hysteresis loop undergoes a bifurcation, leading to the formation of two stable and equivalent but symmetry broken trajectories. Stochastic resonance can be observed by applying an additional sinusoidal signal. [S1063-651X(98)02611-7]

PACS number(s): 05.45.+b, 42.65.Pc, 02.90.+p, 42.55.−f

I. INTRODUCTION

The phenomenon of stochastic resonance has been a subject of intense investigation in recent years and many new areas of applicability have been discovered since the early proposals for its possible relevance in the context of geophysical dynamics [1]. The essence of the idea of stochastic resonance is that a signal can be amplified by letting it influence the dynamics of a system that is either explicitly noisy or has some kind of chaos in it that mimics the presence of noise [2–11]. And resonance occurs when some characteristic internal time scale of the dynamics matches the time period of the applied signal. The most frequently cited example of this phenomenon is the overdamped one-dimensional motion of a particle in a symmetric double-well potential that is subjected to a periodic modulation and a Gaussian white noise. Here the noise controls the time scale of the incoherent thermal tunneling between the two potential wells and the periodic modulation is the external signal that one wants to amplify. Not surprisingly the amplitude of the modulation is taken to be so small that, by itself, the signal cannot induce transitions between the two wells. In this paper we present a mechanism for realizing stochastic resonance in this system when the signal referred to above is very large in amplitude and is actually a part of the dynamics before the signal is added. This is done by exploiting the property of dynamical symmetry breaking of the attractor as the frequency of modulation exceeds a critical value for a fixed value of the amplitude of modulation when this amplitude is higher than the static threshold of dynamical hysteresis. The signal to be amplified in our case is an additional sinusoidal perturbation that, of course, has very small amplitude. In the present study the large amplitude modulation serves the purpose of creating two distinct and stable attractors with broken symmetry. The noise produces tunneling between these two attractors with a characteristic time scale that becomes the relevant internal time scale. This time scale is then made to match the signal time period by tuning the strength of the noise—leading to the observation of stochastic resonance in the process. The primary objective of this paper is to demonstrate the existence of this internal time scale and describe one possible application in the context of stochastic resonance. This paper is organized as follows: In Sec. II we describe the concept of dynamical symmetry breaking when there is no noise. Detection and quantification of dynamical symmetry breaking when noise is present forms the subject of Sec. III. In Sec. IV we demonstrate the application in the context of stochastic resonance. Finally, in Sec. V we present a discussion of the results, point out the directions in which further work needs to be done and some connections to previous works.

II. DYNAMICAL SYMMETRY BREAKING WITHOUT NOISE

The basic model we deal with is given by [12–17]

$$\dot{x}(t)=x-x^3+\alpha\cos\omega t+f(t). \tag{1}$$

This describes the overdamped one-dimensional dynamics of a particle in a bistable potential $U(x)=-x^2/2+x^4/4$. The modulating potential is $-\alpha x\cos\omega t$ and $f(t)$ is the temporally delta correlated random Gaussian noise. On the right hand side of Eq. (1), the term $\alpha\cos\omega t$ normally represents the signal one wants to amplify and the amplitude (α) is much less than the static threshold (α_c) of dynamical hysteresis. Actually α_c equals $2/3\sqrt{3}$ and is the minimum value of α above which the particle can make transitions between the two potential wells around $x=\pm1$ even without the noise. The point of departure in our study is to create a definition of the two stable ''states'' between which noise will eventually induce transitions by including the term $\alpha\cos\omega t$ in the dynamics itself (signal has to be added later on) and taking arbitrary values of α sufficiently greater than α_c. The creation of these two ''states'' results from the following considerations: Consider Eq. (1) without the noise term and, to make simple analytical demonstration possible, take a value of $\alpha\gg1$. Then the asymptotic periodic solution to the damped dynamics is given by

$$x(t)=x_0+\frac{\alpha}{\omega}\sin\omega t \tag{2}$$

*Electronic address: sarkar@jnuniv.ernet.in
†Author to whom correspondence should be addressed. Electronic address: sps@jnuniv.ernet.in

to a very good approximation provided ω is sufficiently high. Substitution of Eq. (2) in Eq. (1) leads to

$$x_0\left(1-\frac{3\alpha^2}{2\omega^2}-x_0^2\right)=0, \tag{3}$$

which means either $x_0=0$ or $x_0=\pm\sqrt{1-3\alpha^2/2\omega^2}$. If $x_0=0$, it corresponds to a situation where the average position of the particle is at the point of reflection symmetry ($x=0$) of the bistable potential. Since in the dynamics described by Eq. (1) there is no preference between positive and negative sides of the point $x=0$, we define an asymptotic solution with average x coordinate equal to zero to be a symmetry conserving one. Since x_0 has to be real, $x_0=0$ is the only possibility when α/ω is greater than $\sqrt{2/3}$. If, however, α/ω is less than $\sqrt{2/3}$ there are two extra real solutions given by $x_0=\pm\sqrt{1-3\alpha^2/2\omega^2}$. Since these two new solutions do not have the average position of the particle at the point $x=0$, we call them symmetry broken ones. Thus, in this paper, the expression ''dynamical symmetry breaking'' is used in the very specific sense of the average position of the particle on the asymptotic periodic solution to the dynamics not corresponding to the point of reflection symmetry of the bistable potential. The full symmetry of Eq. (1) without the noise term is that it is invariant under the joint operations of $x\rightarrow -x$ and $t\rightarrow t+(2n+1)\pi/\omega$ where n is any non-negative integer. Thus any solution should map itself onto another solution under these symmetry transformations. On inspection of the solution given by Eq. (2) it is obvious that this is indeed the case. If $x_0=0$ the solution maps onto itself under these transformations whereas for $x_0=\beta\neq 0$ the mapping is onto a solution with $x_0=-\beta$. Thus, according to our nomenclature, ''dynamical symmetry breaking'' also stands for the degeneracy of the solutions. However, we still have to check the nature of the stability of the solutions given by Eq. (2).

If λ is the factor by which an initial perturbation to the solution grows over one cycle of the modulation over a time period $T(=2\pi/\omega)$, then the stability exponent Ω is defined by $\Omega=\ln\lambda$. From the solution given by Eq. (2) it is easy to show that Ω is given by $(1-3x_0^2-3\alpha^2/2\omega^2)$. Thus in the symmetry broken (SB) phase ($x_0\neq 0$) Ω is given by $\Omega_{SB}=-2(1-\frac{3}{2}\chi^2)$ whereas in the symmetry conserved (SC) phase ($x_0=0$) Ω is given by $\Omega_{SC}=(1-\frac{3}{2}\chi^2)$. Here $\chi=\alpha/\omega$. This immediately shows that when α/ω is less than $\sqrt{2/3}$ the SC solution is unstable whereas the SB solutions are stable. To summarize, for $\alpha/\omega>\sqrt{2/3}$ the only solution that exists is the SC one and it is stable whereas for $\alpha/\omega<\sqrt{2/3}$ the SC solution still exists but is unstable. In this latter domain there are two SB solutions also and they are the stable ones now and hence are the attractors of the dynamics with nonoverlapping domains of attraction. This simple demonstration of the existence of dynamical symmetry breaking (DSB) as well as the fact that only the ratio α/ω (rather than α and ω separately) is the relevant parameter was possible only when α is much greater than unity.

When α is greater than α_c but is not large in magnitude, we have to resort to numerics to demonstrate the existence of DSB. For example, in Fig. 1, we show the mean position ($\langle x\rangle$) of the particle on the attractor, defined by $\langle x\rangle=T^{-1}\int_\tau^{T+\tau}x(t)dt$, as a function of frequency ω, with $\alpha=1.0$. Symmetry breaking appears at a critical frequency ω_c, which is, in general, a function of the value of amplitude (α). This exercise has been carried out for a large range of values of the amplitude above the static threshold α_c for dynamic hysteresis and the data is summarized in Fig. 2. In this diagram, for a given value of the amplitude, as one increases the frequency the hysteresis loop area $A(\alpha,\omega)$, defined as $A(\alpha,\omega)=\alpha\oint x(t)d(\cos\omega t)$ on the periodic attractor, always increases first until it reaches a maximum on reaching the dashed line and then starts decreasing. As one increases the frequency even further DSB appears on reaching the continuous line. Thus, for any (α,ω) point to the right of the continuous line, there are two attractors with distinct domains of attraction. In the large amplitude region, as we have shown analytically, $\omega_c(\alpha)$ should be proportional to α so that the slope of the continuous line should be unity on a log-log plot. This is indeed borne out by the numerical results presented in Fig. 2. What is a little surprising, however, is that this proportionality continues to hold for values of amplitude all the way up to very close to unity even though the derivation given above seems to suggest that the amplitude should be much larger than unity for this property to hold well. In the amplitude range of $\alpha_c<\alpha<\Gamma$ (Γ being very close to 2) there is yet another associated feature of DSB in addition to

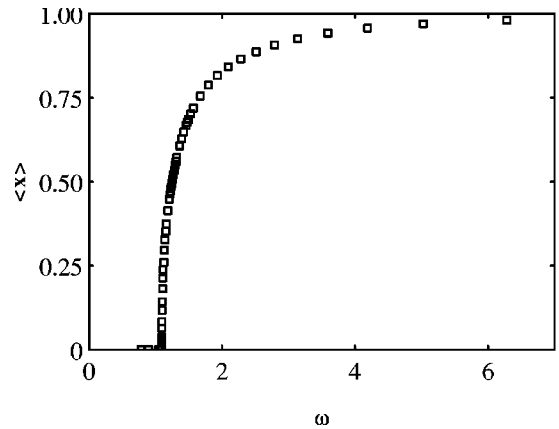

FIG. 1. Mean position of the particle on the attractor $\langle x\rangle$ as a function of frequency ω for $\alpha=1.0$.

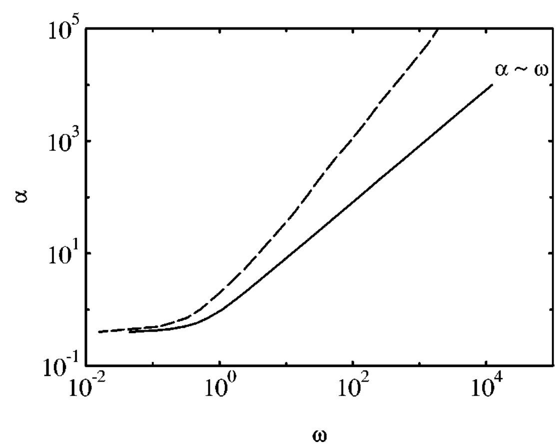

FIG. 2. Phase diagram for the maximal hysteresis loss and the onset of dynamical symmetry breaking in the amplitude-frequency (α-ω) plane.

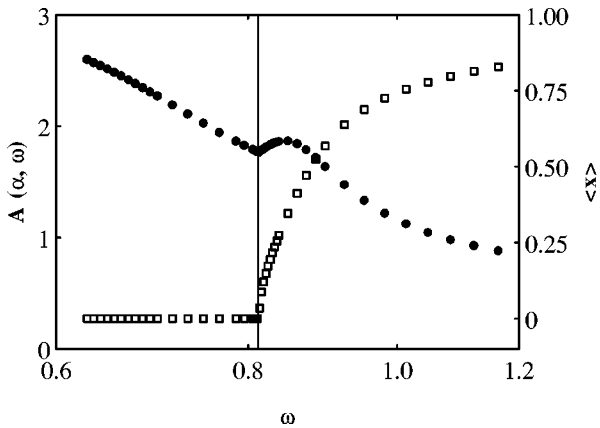

FIG. 3. Hysteresis loop area $A(\alpha,\omega)$ and the mean position $\langle x \rangle$, denoted by the symbols ● and □, respectively, are plotted as a function of frequency (ω) in the same diagram to show that the dip in the loop area takes place exactly at the onset of dynamical symmetry breaking. Here amplitude (α) equals 0.8. The vertical line at the critical frequency is a guide to the eye.

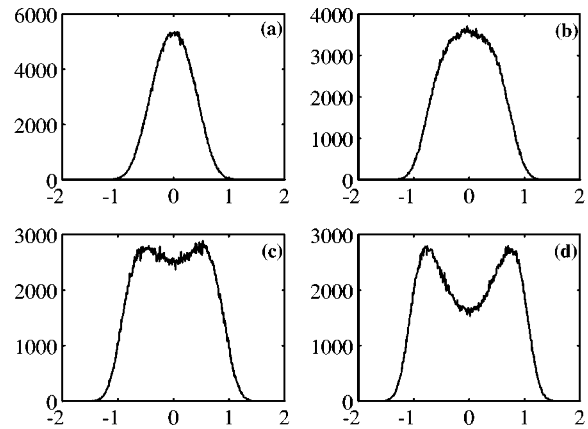

FIG. 4. Histogram for the cycle averaged x coordinate ($\langle x \rangle$) shown for four different values of the time period (T): (a) $T=0.07$, (b) $T=0.055$, (c) $T=0.045$, and (d) $T=0.035$. The values of amplitude (α) and the dimensionless temperature (ϵ) are 100.0 and 0.6, respectively.

the nonvanishing value of the average value of x coordinate. A plot of the area $A(\alpha,\omega)$ of the hysteresis loop as a function of frequency shows a local dip at $\omega=\omega_c(\alpha)$. Figure 3 shows data in support of this observation for $\alpha=0.8$. However, this local dip disappears when α exceeds Γ. Presently, we do not understand this feature at all.

III. DYNAMICAL SYMMETRY BREAKING IN THE PRESENCE OF NOISE

When noise is turned on in Eq. (1), the detection and quantification of DSB immediately become somewhat more complex. The dynamics is stochastic now and the concept of an attractor is no longer applicable. The motivation behind introducing the particular definition of DSB in the noiseless case was that, on each attractor, the particle spends more time on one side of $x=0$ than on the other and the average value of the x coordinate on each attractor is a natural and simple quantifier of the extent of symmetry breaking. This quantification, however, becomes immediately inapplicable when noise is turned on since this average value will be zero. This follows from the fact that the particle will tunnel back and forth between the two potential wells and spend equal time on both (since the characteristics of the noise is independent of the position coordinate). To motivate the definitions that we are going to introduce we begin by considering the noise to be a perturbation. If the parameters of the dynamics are such that, without noise, the attractor is of the symmetry conserved type, then the effect of adding the noise is, roughly speaking, of the following nature: On a time scale of the order of the period of modulation the particle motion is sinusoidal. However, the cycle averaged x coordinate ($\langle x \rangle$) [18] around which this approximately sinusoidal motion takes place keeps on fluctuating due to the presence of the noise—with the most probable value of $\langle x \rangle$ being zero. On the other hand, for parameter values corresponding to symmetry broken dynamics (in the absence of noise), the distribution of the cycle averaged x coordinate will have two separate peaks symmetrically situated around $x=0$.

On the basis of the intuitive picture presented in the previous paragraph the procedure we adopt to detect the presence of DSB when noise is applied is the following: From the $x(t)$ data generated via integration of the Langevin equation, compute the average value of the x coordinate during successive slabs of duration T (the time period of modulation) on the time axis. Generate a histogram for these values of $\langle x \rangle$. Then study this sequence of histograms as a function of frequency. At the onset of DSB the histogram for $\langle x \rangle$ will undergo a transition from unimodal to bimodal around $\langle x \rangle=0$. That this is indeed the case is illustrated in Figs. 4(a) to 4(d) for $\alpha=100.0$. The dimensionless temperature (ϵ), defined as the ratio of the Boltzmann constant times the temperature divided by the height of the barrier separating the two potential minima, equals 0.6 here. The actual quantification of the extent of DSB can be done in several ways. For example, one can (i) take the positions of the two peaks or (ii) try to fit some function to the histogram that will contain the extent of symmetry breaking as a parameter. Here we adopt the second procedure in which the distribution $P(\langle x \rangle)$ is fitted as a sum of two Gaussians with the same width σ but centered at $\pm\delta$. Weights of the two Gaussians, w_1 and w_2, are allowed to be different in the fitting procedure to accommodate the possibility that the run may not have been long enough for the particle to have spent equal time in the two wells. δ is taken to be the measure of symmetry breaking. Clearly this procedure is somewhat ad hoc since we have not proved that the functional form of the histogram is actually of the type assumed. In any case Fig. 5 shows a plot of $\delta(\omega)$ thus calculated (in analogy to Fig. 1 for the noiseless case).

Next, we address the issue of temporal correlations in the discrete time series for $\langle x \rangle$. As is standard, the autocorrelation function is calculated and Fig. 6 presents some typical data both on linear and semilogarithmic formats. The fact that the semilogarithmic plot of Fig. 6(b) is a straight line to a very good approximation means that there is indeed a very well defined and unique time scale that corresponds to interwell tunneling. This time scale $\tau(\omega)$ is the analog of the conventional Kramers' time scale that one gets in the absence of the modulation and it describes and quantifies temporal correlations on a time scale much longer than the period of modulation—at least in the strongly symmetry

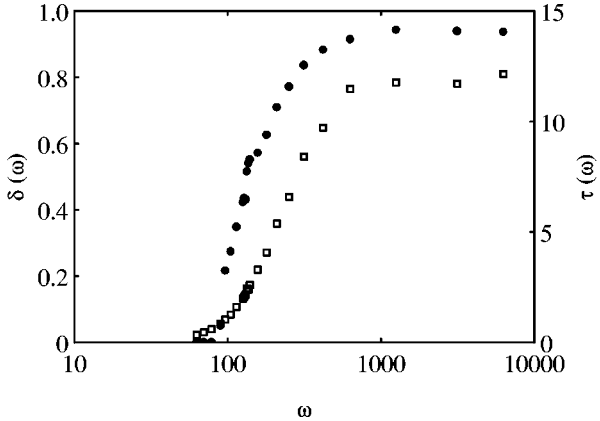

FIG. 5. Autocorrelation time $[\tau(\omega)]$ for $\langle x\rangle$ and the extent of symmetry breaking $[\delta(\omega)]$ are plotted as a function of frequency (ω) with $\epsilon=0.6$ and $\alpha=100.0$. Symbols for $\delta(\omega)$ and $\tau(\omega)$ are ● and □, respectively.

broken regime and for large amplitudes of modulation. In Fig. 5 we plot $\tau(\omega)$ also. We wish to emphasize that the existence of this new time scale and its dependence on frequency is the central point of this paper. In Fig. 5 notice that as frequency becomes very large δ saturates to a value close to unity and $\tau(\omega)$ saturates (apart from numerical uncertainties) to a constant τ_0. The reason for this saturation is that as the frequency becomes very large the modulation of the form $\alpha \cos \omega t$ averages itself out to zero on a time scale larger than a few times the period of modulation more and more effectively. In the limit of ω going to infinity this autocancellation effect is complete and the modulation drops out of the dynamics on this larger time scale. Thus δ and τ will assume values that one gets from Eq. (1) without the modulation term. The fact that the limiting value of δ should be close to unity follows from the form of the bistable potential whereas the autocorrelation time for the modulation free case can be computed separately and it turns out to be same as τ_0. This has been checked for several values of the dimensionless temperature (ϵ) and amplitude (α) and τ_0 is found to depend on temperature only—as is expected. For example, in Fig. 5, the dimensionless temperature is 0.6 and the expected saturation value of $\tau(\omega)$, as found from the separate Langevin integration of the modulation free case, is 11.6. To within numerical uncertainties this agrees very well with the data for $\tau(\omega)$ in the large ω limit. The other important feature that has emerged out of our numerical computations is that both τ and δ seem to depend only on the ratio α/ω for large amplitudes for a given value of the temperature. We must point out that the procedure that we have adopted for detection and quantification of DSB works only when noise can be treated as a relatively weak perturbation, i.e., the period of modulation at the symmetry breaking transition is much less than the Kramers' tunneling time for the unperturbed case. For a fixed temperature, this will be more and more accurately valid as the amplitude (α) increases since the critical frequency will also then keep on increasing. When noise cannot be treated as a weak perturbation the situation is much more difficult and we have no simple analog to the abovementioned prescription.

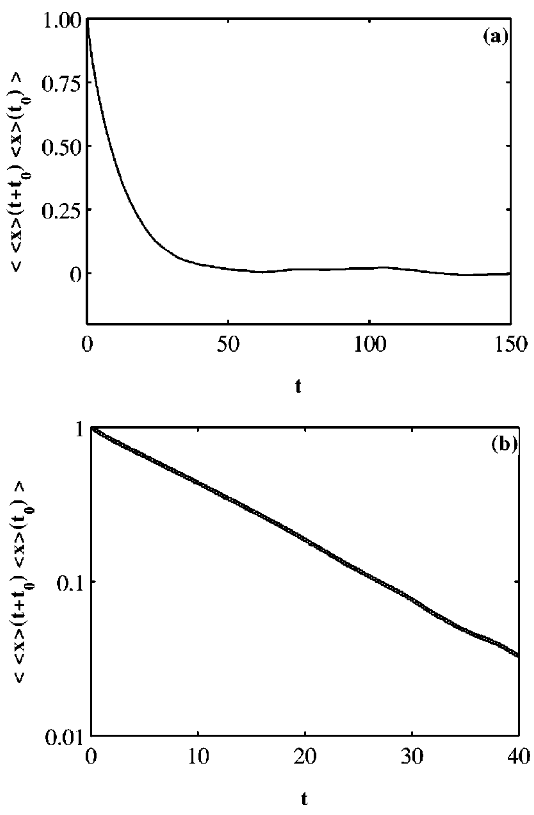

FIG. 6. Normalized autocorrelation function $C(t)=\langle\langle x\rangle(t+t_0)\times\langle x\rangle(t_0)\rangle/\langle\langle x\rangle^2\rangle(t)$ plotted in (a) linear format and (b) semilogarithmic format for the parameter values $\alpha=10.0$, $T=0.25$, and $\epsilon=0.4$. Here time is understood to be the cycle number multiplied by the period (T) of modulation.

IV. STOCHASTIC RESONANCE

It should be apparent from the discussion in the previous section that the temporal properties of the cycle averaged x coordinate in the solution of Eq. (1) are very similar, if not qualitatively identical, to the real time solution of the same equation without the modulation—although with a frequency dependent time scale of $\tau(\omega)$ that is analogous to (but different from) the Kramers' time for thermal tunneling in the modulation free case. This is true provided the amplitude (α) is sufficiently high for the given value of the temperature. Thus we have all the ingredients needed for observing stochastic resonance (SR)—although now corresponding to the dynamics of the cycle averaged x coordinate and with $\tau(\omega)$ playing the role of the internal time scale. Since $\tau(\omega)$ depends on amplitude (α) and frequency (ω) also in addition to the temperature we have more control parameters to tune the internal time scale. Once this is noted, it is straightforward to observe SR by applying a signal by way of adding a term of the form $\gamma \cos \omega' t$ to the right-hand side of Eq. (1). For example, in Fig. 7 we show this effect by plotting the response as a function of temperature for a fixed signal term. It is to be noted that the response plotted here is not the signal-to-noise ratio, as is more conventional. Rather we simply take the time series for the cycle averaged x coordinate and compute the power in the first harmonic corresponding to the signal frequency. For the purpose of demonstration, which is our goal here, it is adequate. We have made a systematic

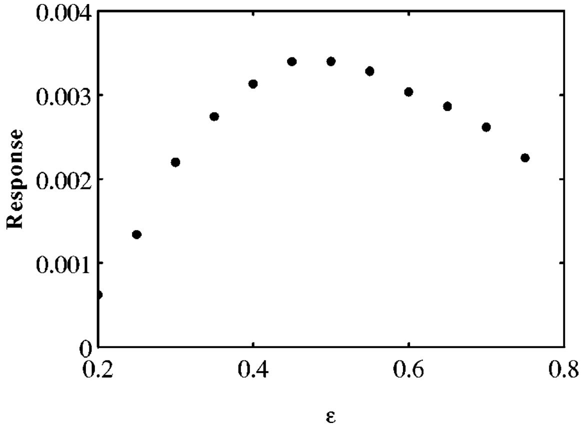

FIG. 7. Response is plotted as a function of the dimensionless temperature (ϵ) with the amplitude and the time period of the signal equal to 0.03 each.

study of the temperature at which the response peaks as a function of the frequency of the signal to be amplified and it agrees with the understanding that already exists in the literature for the more conventional (modulation free) case. Also the power in the response is found to be proportional to the square of the signal amplitude (γ)—as is expected.

V. DISCUSSION

In this paper we have reported on the phenomenon of dynamical symmetry breaking in a modulated bistable potential well in some detail and have demonstrated the possibility of its application in the context of stochastic resonance. It should be possible to realize this particular application through experiments, especially using analog simulation techniques [9]. In this paper we have deliberately avoided using the language of switching to describe the dynamics since it becomes quite meaningless to do so at the high values of amplitude and frequency that we are primarily interested in. The hysteresis loop looks almost elliptical in this parameter domain. But we should point out that what is described in the literature as the presence or absence of deterministic switching at lower values of the amplitude does smoothly evolve into the concept of dynamical symmetry breaking although they do not quite mean the same thing. There are some issues arising out of our results that need to be addressed. (i) The qualitative nature of the (discrete) time evolution of the cycle averaged x coordinate in the dynamics of Eq. (1) seems to be very similar to the (continuous) time evolution of $x(t)$ in Eq. (1) without the modulation term. As we have shown here, there is a very well defined interwell tunneling time that can be thought of as a pseudo-Kramers' time. Is there is a quantitatively precise mapping between these two problems on a time scale sufficiently longer than the period of modulation? That will make it possible to have an analytical derivation of the tunneling time $\tau(\omega)$. (ii) It is straightforward to extend the model of the single damped oscillator to that of a coupled lattice of such oscillators. In addition to the possibility of enriching the phenomena that are already known in these problems, including array enhanced stochastic resonance, this can serve as a more microscopic model exhibiting dynamical phase transitions [19–28]. In the context of this last problem it will obviously be necessary to properly define and quantify symmetry breaking. We hope that the discussion presented in Sec. III of this paper will serve as a prelude to that.

ACKNOWLEDGMENT

One of the authors (D.B.) acknowledges the University Grants Commission of India for financial support.

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